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THE IMPOSSIBILITY OF EINSTEIN FIELDS IMMERSED IN FLAT SPACE OF FIVE DIMENSIONS.

By Edward Kasner.

By the theory of quadratic differential forms it follows that a general riemannien manifold of m dimensions can always be regarded as immersed in some flat space of n dimensions, where n does not exceed $\frac{1}{2}m(m+1)$. Thus if $m = 4$, as in the Einstein theory, the form

$$ds^2 = \Sigma g_{ik} dx_i dx_k \qquad (i, k = 1, 2, 3, 4)$$

can be immersed in an n-flat, where the possible values of n are 4, 5, 6, 7, 8, 9, 10.

If now we require the manifold to obey Einstein's equations of gravitation, $G_{ik} = 0$, the question arises which of these values of n are actually realizable. The case $n = 4$ is trivial, since then the curvature vanishes and there is no permanent gravitation. *We wish to show now that the case $n = 5$ is impossible; that is, no Einstein manifold can be regarded as imbedded in a five-flat.*

In a flat space of five dimensions let rectangular coordinates be denoted by x_1, x_2, x_3, x_4, w. Then any four-dimensional manifold M in that space may be defined by a single finite equation

$$w = f(x_1, x_2, x_3, x_4). \tag{1}$$

The element of length of M is

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dw^2,$$

where

$$dw = f_i dx_i, \qquad \left(f_i = \frac{\partial f}{\partial x_i}\right),$$

is the total differential, summation with respect to the repeated index i being understood as usual. This can be written

$$ds^2 = g_{ik} dx_i dx_k, \tag{2}$$

where

$$g_{11} = 1 + f_1^2, \qquad g_{12} = f_1 f_2, \qquad \text{etc.} \tag{3}$$

The determinant of (2) is

$$g = \begin{vmatrix} 1+f_1^2 & f_1f_2 & f_1f_3 & f_1f_4 \\ f_1f_2 & 1+f_2^2 & f_2f_3 & f_2f_4 \\ f_1f_3 & f_2f_3 & 1+f_3^2 & f_3f_4 \\ f_1f_4 & f_2f_4 & f_3f_4 & 1+f_4^2 \end{vmatrix}, \tag{4'}$$

which reduces exactly to

$$g = 1 + f_1^2 + f_2^2 + f_3^2 + f_4^2. \tag{4}$$

The minors of (4′) also simplify, giving

$$\begin{aligned} g^{11} &= \frac{1 + f_2^2 + f_3^2 + f_4^2}{g}, \\ g^{12} &= \frac{f_1 f_2}{g}, \qquad \text{etc.} \end{aligned} \tag{5}$$

Our problem is to find the function f so that the ten functions (3) shall satisfy the ten gravitational equations* $G_{ik} = 0$.

CALCULATION OF THE CHRISTOFFEL SYMBOLS.

We now calculate the Christoffel symbols (of the second kind)

$$\{\alpha\beta, \gamma\} = \tfrac{1}{2} g^{\gamma\epsilon}(g_{\alpha\epsilon,\,\beta} + g_{\beta\epsilon,\,\alpha} - g_{\alpha\beta,\,\epsilon}). \tag{6}$$

From (3) we find the derivatives

$$g_{ik,\,j} = (f_i f_k)_j = f_i f_{kj} + f_k f_{ij},$$

this formula holding whether the subscripts are unequal or equal. Substituting in (6), we find

$$\{\alpha\beta, \gamma\} = f_{\alpha\beta} f_\epsilon g^{\gamma\epsilon}. \tag{6′}$$

Here the second derivative $f_{\alpha\beta}$ appears as a factor, and it remains to carry out the summation $f_\epsilon g^{\gamma\epsilon}$ involving only first derivatives of f. If the variable index ϵ takes the value γ, we have

$$f_\gamma g^{\gamma\gamma} = f_\gamma \frac{1 + \Sigma' f_{\epsilon'} f_{\epsilon'}}{g} \qquad \text{(no summation on left)},$$

where the prime indicates that ϵ' is to vary through the three values different from γ. For the three values of ϵ different from γ, we have

$$\Sigma' f_{\epsilon'} g^{\gamma\epsilon'} = f_\gamma \frac{\Sigma' f_{\epsilon'} f_{\epsilon'}}{g}.$$

Therefore

$$f_\epsilon g^{\gamma\epsilon} = f_\gamma \frac{1 + 2\Sigma' f_{\epsilon'} f_{\epsilon'}}{g}.$$

Hence the final value of our symbol is

$$\{\alpha\beta, \gamma\} = \frac{f_{\alpha\beta} f_\gamma P_\gamma}{g}, \tag{7}$$

where

$$P_\gamma = 1 + 2\Sigma' f_{\epsilon'} f_{\epsilon'} = 2g - 1 - 2f_\gamma^2. \tag{7′}$$

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* For the notation see Weyl or Eddington, or § 1 of our paper "Einstein's theory of gravitation: determination of the field by light signals," this JOURNAL, vol. 43 (1921), p. 20.

CALCULATION OF THE EINSTEIN TENSOR G_{ik}.

Take first the case of unlike indices,

$$G_{12} = \{1\alpha, \beta\}\{2\beta, \alpha\} - \{12, \alpha\}L_\alpha - \{12, \alpha\}_\alpha + L_{12}, \tag{8}$$

where $L = \frac{1}{2} \log g$. We now simplify the rest of the discussion by taking our axes in the original five-flat so the w-axis is normal to the four-spread (1) and so that the axes x_1, x_2, x_3, x_4 agree with the principal directions of the spread at the selected point, which we select as origin. This means that the partial derivatives of first order f_1, f_2, f_3, f_4 all vanish, and that

$$f_{12} = f_{13} = f_{14} = f_{23} = f_{24} = f_{34} = 0.$$

The four-spread (1) may therefore be written in power series form

$$w = \tfrac{1}{2}(k_1x_1^2 + k_2x_2^2 + k_3x_3^2 + k_4x_4^2) + \text{higher terms.} \tag{9}$$

Thus at the origin

$$f_{11} = k_1, \qquad f_{22} = k_2, \qquad f_{33} = k_3, \qquad f_{44} = k_4,$$

these values defining the principal curvatures. We have also $g = 1$, so that $L = 0$, at the origin.

For this special choice of axes we see from (7) that all the Christoffel symbols vanish. This throws out the first and second terms of (8). The third and fourth terms involve second derivatives of f, but we find that both terms vanish. Hence G_{12}, and of course the other five combinations with unlike subscripts, vanish identically.

Take now the case of like indices

$$G_{11} = \{1\alpha, \beta\}\{1\beta, \alpha\} - \{11, \alpha\}L_\alpha - \{11, \alpha\}_\alpha + L_{11}. \tag{10}$$

Here the first and second terms vanish. The third term is found to be

$$- k_1(k_1 + k_2 + k_3 + k_4).$$

The last term reduces to

$$k_1^2.$$

Hence*

$$G_{11} = - k_1(k_2 + k_3 + k_4). \tag{11}$$

Thus the conditions that the four-spread (9) *shall be of the Einstein type (that is, obey* $G_{ik} = 0$) *are*

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* From this we may find the known formula for scalar curvature

$$G = g^{ik}G_{ik} = -2(k_1k_2 + k_1k_3 + k_1k_4 + k_2k_3 + k_2k_4 + k_3k_4).$$

Cf. Eddington, "Report," p. 75, or Page, *Trans. Connecticut Acad*,, Vol. 23 (1920), p. 408. We may also connect with the elegant interpretations of the tensor G_{ik} given by Herglotz, *Leipziger Berichte*, 1916 (not accessible), and Vermeil, *Göttinger Berichte*, 1917.

$$
\begin{aligned}
k_1(k_2 + k_3 + k_4) &= 0,\\
k_2(k_3 + k_4 + k_1) &= 0,\\
k_3(k_4 + k_1 + k_2) &= 0,\\
k_4(k_1 + k_2 + k_3) &= 0.
\end{aligned} \tag{12}
$$

On the other hand the conditions that (9) shall be flat (euclidean or homoloidal) are

$$k_1k_2 = k_1k_3 = k_1k_4 = k_2k_3 = k_2k_4 = k_3k_4 = 0. \tag{13}$$

The two sets (12) and (13) are easily seen to be equivalent, each meaning that at least three of the four quantities k_1, k_2, k_3, k_4 must vanish.

This shows that in a flat space of five dimensions we cannot construct four-dimensional manifolds (other than the trivial euclidean type) which obey Einstein's equations. *That is, an Einstein spread representing a permanent gravitational field can never be regarded as immersed in a five-flat.*

In flat space of six dimensions, actual Einstein manifolds exist; in particular, the solar field discussed in the next paper.

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